# Stat 444
# Advanced Long-term Acturial Math

## Lecture 4: Profit Analysis

# Annual profit

We can consider the emergence of profits for a block of policies by comparing actual experience to expected experience.

If actual experience precisely mirrors assumptions, then there won't be a profit or loss in any year, but this is very unlikely (and sometimes impossible).

We will define the **profit** for a block of policies for policy year $k$ as:

(Reserve at time $k-1$ plus net cash flows at beginning of year, accumulated to the end of the year) - (Reserve at time $k$ plus net cash flows at end of year)

A positive value of this quantity will be a profit; If this quantity is negative, it will be a loss.

# Example

We sell fully discrete whole life policies with \$100,000 death benefits to 100 independent 50-year-olds.

Premium and Policy Value Basis:

- Mortality is given by SULT
- $i = 5\%$
- The only expenses are 5% of premiums

The actual experience for the first two years is:

| | **First Year** | **Second Year** |
|---|---|---|
| Deaths | 1 | 0 |
| Expenses (as a % of premium) | 5.5% | 4.5% |
| Interest Earned | 6% | 4% |

Calculate the profit for the first two years. [-86442.61, 11303.48]

# Profit by source

For a given year, we can break down the profit into its source components.

For each source, the profit is the difference between the actual and expected cash flows attributable to that source, valued at the end of the year.

To avoid double counting, we use assumed (expected) values for sources not yet considered and actual values for sources already considered.

The order in which we consider the sources can impact the attribution of profit to sources, though these differences are usually small. For any order, the sum of the profits by source should equal the total profit for the year.

# Profit by source

Mortality

$$(\text{Deaths}_{\text{expected}} - \text{Deaths}_{\text{actual}})(S + E_T - {}_tV)$$

Interest

$$(n\ {}_{t-1}V + nP - E)(i_{\text{actual}} - i_{\text{expected}})$$

Expenses

$$(E_{\text{expected}} - E_{\text{actual}})(1 + i) + (E_{T:\text{expected}} - E_{T:\text{actual}})\mathit{Deaths}$$

The blue terms change from expected to actual

| Order | Expected | Actual |
|---|---|---|
| IME | $E$, $E_T$ | $i$, Deaths |
| IEM | $E$, Deaths | $i$, $E_T$ |
| EIM | $i$, Deaths | $E$, $E_T$ |
| EMI | $i$, Deaths | $E$, $E_T$ |
| MIE | $E$, $E_T$ | $i$, Deaths |
| MEI | $i$, $E_T$ | $E$, Deaths |

# Example (continued)

For the previous example (in year 2), we can break down the profit by source (up to rounding error) — we see that the order slightly affects the decomposition:

| **Source** | **Profit** |
|---|---|
| Mortality | 12,895.22 |
| Expenses | 608.37 |
| Interest | -2,143.58 |
| Total | 11,360.01 |

| **Source** | **Profit** |
|---|---|
| Interest | -2,137.79 |
| Mortality | 12,895.22 |
| Expenses | 602.58 |
| Total | 11,360.01 |

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**Note**: The difference between the death benefit payable (plus any associated expenses) and the reserve at the end of the year is sometimes known as the **Net Amount at Risk (NAR)**; this is the amount of the mortality risk associated with a policy or block of policies for the insurer for the year.

# Asset shares

We've seen that the policy value represents the amount, per policy, that the insurer needs to have on hand in to be able to expect to — combined with future premiums — cover expected future benefits.

Similarly, the amount that the insurer *actually* does have on hand per policy at time $t$ is called the **asset share** and is denoted $AS_t$.

It's usually calculated by considering a large number of identical policies and imagining that there's a dedicated fund set up for these policies:

- The fund starts with $0 at the time of issue.
- Premiums are paid into the fund; expenses and claims are paid out of the fund, and the fund earns interest.
- Then the asset share for each policy still in force at time $t$ is

$$AS_t = \frac{\text{amount in fund at time } t}{\text{number of policies in force at time } t}$$

# Asset share example

Consider a block of 1,000 identical (and independent) 20-year term policies issued to 40-year-olds, each with $\$500,000$ death benefit payable at the end of the year and gross premium of $\$1,100$ payable at the beginning of each year the policy is in force.

Suppose that the insurance company has the following experience over the first three years, and that all expenses are paid at the beginning of the year:

| Year | Expenses | Interest Rate Earned | Deaths |
|---|---|---|---|
| 1 | $\$242,000$ | 5.1% | 1 |
| 2 | $\$168,500$ | 4.7% | 2 |
| 3 | $\$165,000$ | 4.6% | 1 |

# Asset share example (continued)

We can calculate the asset share at times 1, 2, and 3:

| Year | End of Year Fund Amount | Survivors | $AS_t$ |
|---|---|---|---|
| 1 | $401,758 | 999 | $402.16 |
| 2 | $394,769 | 997 | $395.96 |
| 3 | $887,487 | 996 | $891.05 |

The amount in the fund at the end of the first year would be calculated as:

$$[0 + (1,000)(1,100) - 242,000](1.051) - (1)(500,000) = 401,758$$

The asset share at time 1 would be calculated as:

$$AS_t = \frac{401,758}{999} = 402.16$$

# Profit Testing

Profit testing is commonly done by actuaries in life insurance companies. It's useful for a number of reasons:

- Setting premium rates or testing various premium levels.
- Testing the sensitivity of different pricing assumptions or bases.
- Determining the impact of different reserving methods or bases.
- Analyzing levels of surplus and profit for a product.
- Performing stress tests or scenario tests for a product.

In practice, profit testing is often an iterative process. Here we'll think of premiums as an input, rather than an output.

# Profit Test Basis and Notation

The **profit test basis** consists of the assumptions used when doing the profit test.

For policy year $k$, for a policy in force at time $k-1$, we denote:

${}_{k-1}V$ is the reserve at the beginning of the year

$P_k$ is the premium paid at the start of the year

$E_k$ is the expenses incurred at the start of year $k$, for $k = 1, 2, \ldots$

Note that in our profit tests, we separate the **acquisition expenses** or **pre-contract expenses** from the other first-year expenses; the pre-contract expenses are denoted by $E_0$

# Profit Test Basis and Notation (continued)

For policy year $k$, for a policy in force at time $k-1$, we denote:

$I_k$ is the interest earned in year $k$ from the assets held at the start of the year

$$I_k = i_k \left( {}_{k-1}V + P_k - E_k \right)$$

$EDB_k$ is the expected death benefit payment at time $k$

$$EDB_k = DB_k \, q_{x+k-1}$$

$E\,{}_kV$ is the expected reserve at the end of the year

$$E\,{}_kV = {}_kV \, p_{x+k-1}$$

$Pr_k$ is the expected profit emerging for year $k$ (at time $k$)

# Profit Vector

Then the expected profit emerging at time $k$ for a policy in force at time $k-1$ is given by

$$Pr_k = {}_{k-1}V + P_k - E_k + I_k - EDB_k - E_k V$$

The **profit vector** for the policy is given by

$$\boldsymbol{Pr} = \begin{pmatrix} Pr_0 \\ Pr_1 \\ \vdots \\ Pr_n \end{pmatrix}$$

# Profit Signature

Recall that $Pr_k$ is the expected profit at time $k$ for a policy in force at time $k-1$. Thus, it's a conditional measure of expected profit for year $k$. We define another quantity

$$\Pi_k = {}_{k-1}p_x \, Pr_k, \quad k = 1, 2, \ldots$$

which is the unconditional expected profit for year $k$, for a policy issued to $(x)$.

The **profit signature** for the policy is given by

$$\mathbf{\Pi} = \begin{pmatrix} \Pi_0 \\ \Pi_1 \\ \vdots \\ \Pi_n \end{pmatrix}$$

where we define $\Pi_0 = Pr_0$.

# Profit Testing Example from AMLCR

Consider a \$100,000 fully discrete 10-year term insurance issued to (60). The profit test basis is:

| | |
|---|---|
| Interest: | 5.5% per year |
| Pre-contract expenses: | \$400 per year + 20% of the first premium |
| Renewal expenses: | 3.5% of premiums, including the first premium |
| Mortality: | $q_{60+k} = 0.01 + 0.001k, \quad k = 0, 1, \ldots, 9$ |
| Gross Premium: | \$1,500 per year |
| Reserves: | (see next slide) |

# Profit Testing Example from AMLCR (continued)

The reserves we'll use for the profit test are the Net premium policy values computed using the following policy value basis:

Interest: 4% per year

Mortality: $q_{60+k} = 0.011 + 0.001k, \quad k = 0, 1, \ldots, 9$

Note that both mortality and interest are different in the reserve basis than in the profit test basis. The resulting reserve values are:

| $k$ | ${}_kV$ | $k$ | ${}_kV$ |
|---|---|---|---|
| 0 | 0.00 | 5 | 1,219.94 |
| 1 | 410.05 | 6 | 1,193.37 |
| 2 | 740.88 | 7 | 1,064.74 |
| 3 | 988.90 | 8 | 827.76 |
| 4 | 1,150.10 | 9 | 475.45 |

# Profit Testing Example from AMLCR (continued)

It's often convenient to arrange the profit test results in a table.

| $k$ | ${}_{k-1}V$ | $P_k$ | $E_k$ | $I_k$ | $EDB_k$ | $E\,{}_kV$ | $Pr_k$ |
|---|---|---|---|---|---|---|---|
| 0 | | | 700.00 | | | | -700.00 |
| 1 | 0.00 | 1,500 | 52.50 | 79.61 | 1,000 | 405.95 | 121.16 |
| 2 | 410.05 | 1,500 | 52.50 | 102.17 | 1,100 | 732.73 | 126.99 |
| 3 | 740.88 | 1,500 | 52.50 | 120.36 | 1,200 | 977.04 | 131.70 |
| ⋮ | | | | | | | |
| 9 | 827.76 | 1,500 | 52.50 | 125.14 | 1,800 | 466.89 | 133.52 |
| 10 | 475.45 | 1,500 | 52.50 | 105.76 | 1,900 | 0.00 | 128.71 |

Table: Excerpt of Table 13.3 in AMLCR3e

And the profit signature is

$$\mathbf{\Pi} = (-700.00, 121.17, 125.72, \ldots, 113.37)'$$

The **Internal Rate of Return (IRR)** is the int. rate $j$ such that

$$\sum_{k=0}^{n} \Pi_k \, v_j^k = 0$$

Usually, we some sort of technological assistance to get a number for $j$, as the above equation is an $n^{th}$ degree polynominal in $j$.

Typically, if the IRR exceeds a predesignated threshhold (sometimes called the cost of capital or hurdle rate), then the project is sufficiently profitable. We will assume a hurdle rate of 10% for this example.

Pros

- Simple to calculate, understand, and interpret
- Works well for a typical profit signature $(-, +, +, \ldots)$

Cons

- Not guaranteed existence or uniqueness of a (real) solution
- No idea of the magnitude of the profits

# Profit Measures — NPV

The **Net Present Value (NPV)** is the PV of the profit signature:

$$NPV = \sum_{k=0}^{n} \Pi_k \, v_h^k$$

where the discounting is done at the hurdle rate.

If the NPV is positive, then the project is profitable.

| Pros | Cons |
|---|---|
| • Gives a PV dollar amount | • No sense of how much cash flow or investment is needed to generate that PV |

We can also define the **partial net present value** as

$$NPV(t) = \sum_{k=0}^{t} \Pi_k \, v_h^k$$

# Profit Measures — Profit Margin

The **Profit Margin** is the NPV expressed as a proportion of the EPV of the premiums, where all discounting is done at the hurdle rate:

$$\text{Profit Margin} = \frac{\text{NPV}}{\text{EPV(Premiums)}}$$

Pros

- Commonly used in life insurance and well-understood
- Easy to calculate and is largely comparable among products

Cons

- Not much information about timing of profits

# Profit Measures — DPP

The **Discounted Payback Period (DPP)** or **Break-even Period** is the first time at which the partial NPV is positive, i.e., DPP is the minimum value of $m$ such that $NPV(m) > 0$, where the discounting is done at the hurdle rate.

**Pros**

- Tells when the insurer expects to begin to be profitable
- Works well for a typical profit signature $(-, +, +, \ldots)$
- Simple to calculate, understand, and interpret

**Cons**

- No information about IRR or overall profitability
- No idea of the magnitude or pattern of the profits
- Very poor as a standalone measure

# Using Profit Testing to Set Premium Rates

Having done the profit test and computed the resulting profit measures, we may find that the profitability isn't sufficient.

- In some very simple cases, it may be possible to solve numerically for the premium level that results in the product meeting all of the profitability criteria.
- More often the process will be iterative:
  - Start with a set of premiums and other assumptions
  - Do the profit test and calculate the profit measures
  - Adjust the premium levels and possibly other assumptions
  - Redo the profit test and recalulate the profit measures
  - Repeat...

In the previous example, what gross premium would we need to charge in order to achieve a 5% profit margin, holding everything else constant? (Answer: 1,572.55)

# Impact of Reserves on Profitability

Reserve levels impact the timing of the emergence of profits: if we increase our reserves, our profits will be lower in the early years when the reserve is building, but higher in the later years when we're releasing reserves.

- This in turn effects the present value of profits.

In general, when the hurdle rate, i.e., the rate that the product needs to earn, is greater than the actual interest rate earned on assets (which is typical), lower reserves will result in greater profitability for the insurer and vice-versa.

# Impact of Reserves on Profitability — Example

To see the impact of reserves on profitability, we'll change the reserves in our previous example and redo our profit test. We'll look at two alternate reserve scenarios: one in which we strengthen the reserve basis, and one where we hold no reserves.

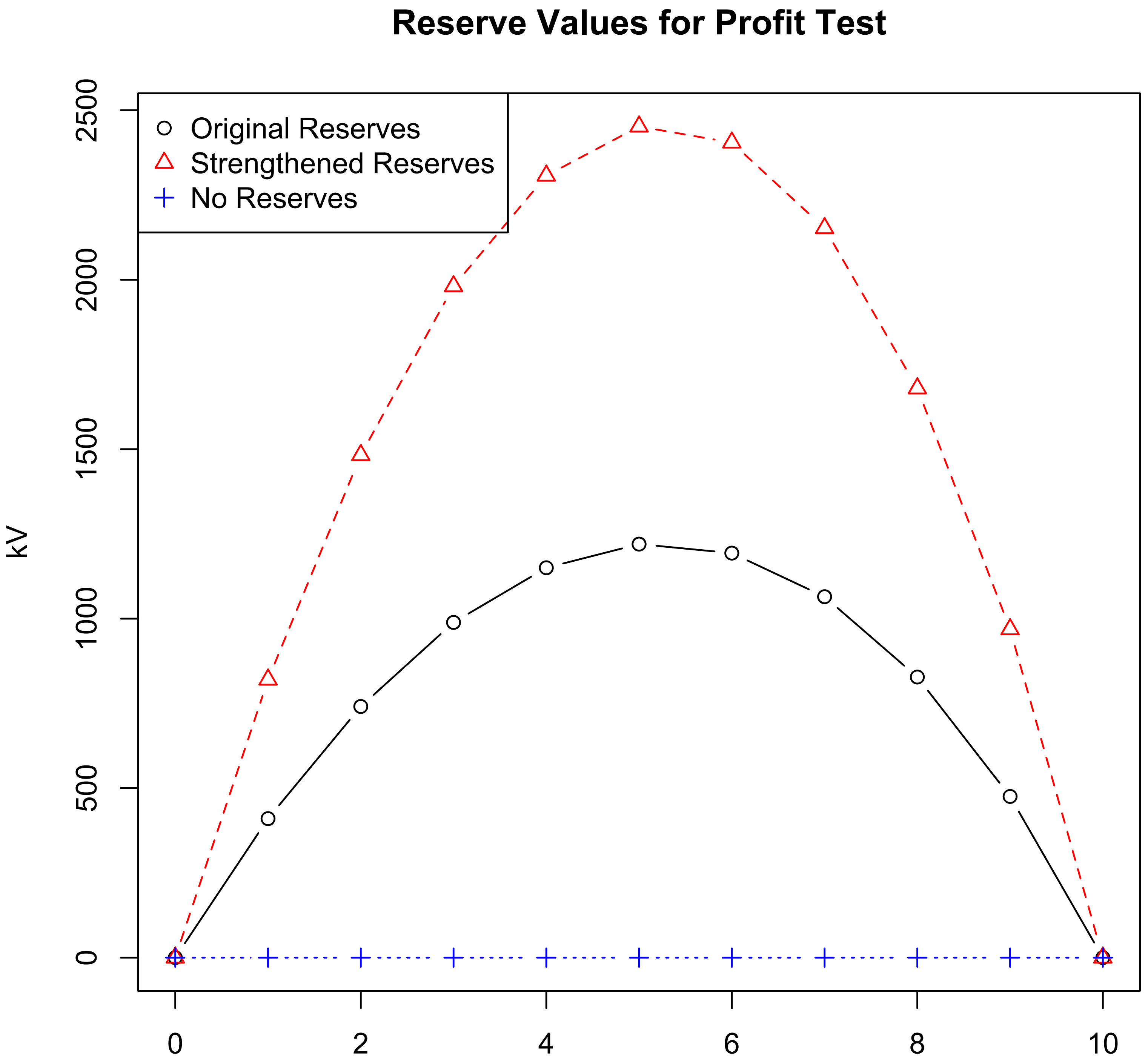
Alternate Reserve Scenario 1: Strengthened Reserve Basis

| | |
|---|---|
| Reserve Method: | Net Premium Policy Value |
| Interest: | 3% per year |
| Mortality: | $q_{60+k} = 0.022 + 0.002k, \quad k = 0, 1, \ldots, 9$ |

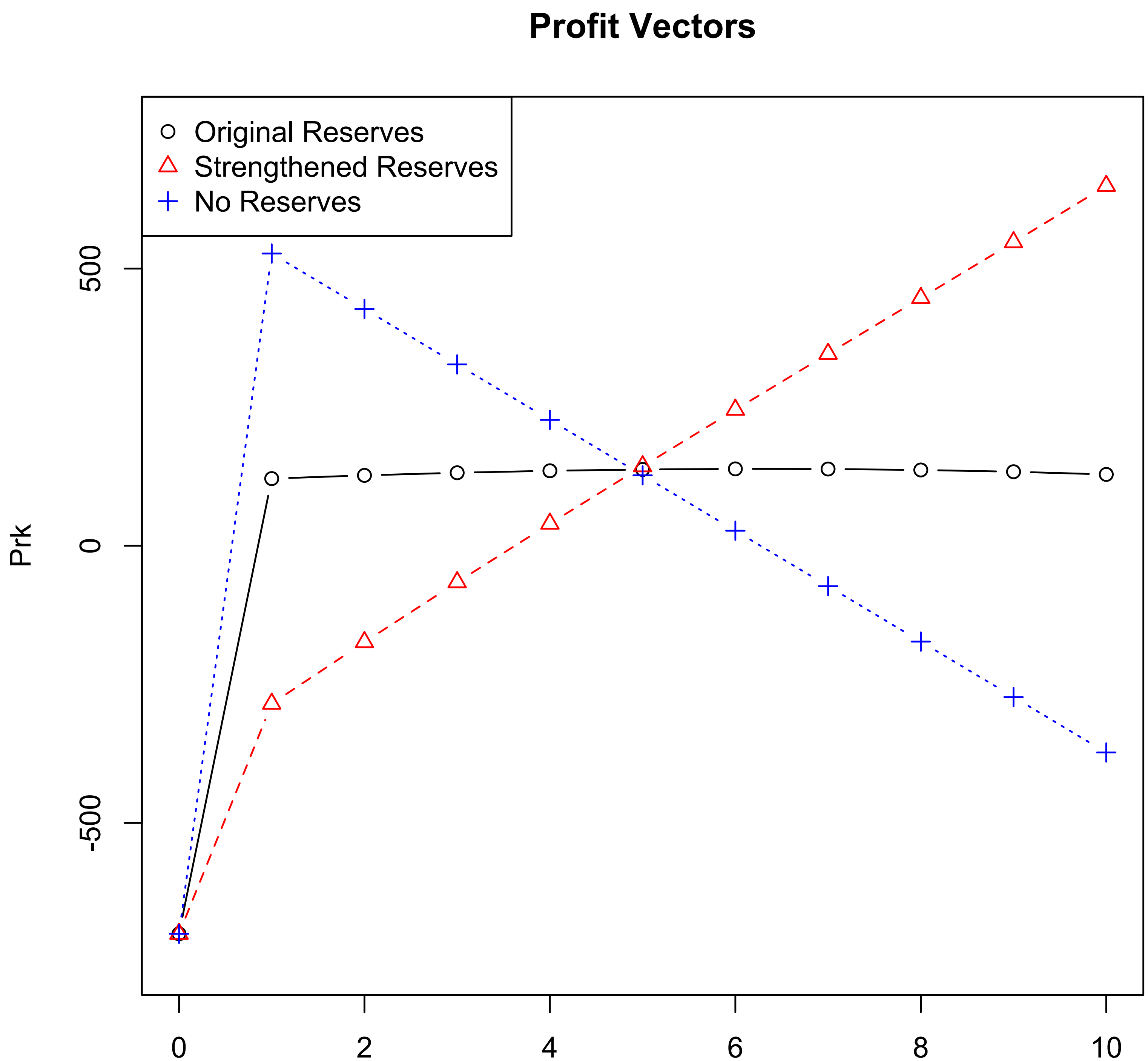
Alternate Reserve Scenario 2: Hold No Reserves

| | |
|---|---|
| Reserve Method: | Set ${}_kV = 0$ for all $k$ |

# Impact of Reserves on Profitability — Example



# Impact of Reserves on Profitability — Example



# Impact of Reserves on Profitability — Example

We can see that the different reserves result in different profit vectors, which leads to different values for our profit measures:

| Scenario | NPV | IRR | DPP | Profit Margin |
|---|---|---|---|---|
| Strengthened Reserves | -124.23 | 8.3% | $\infty$ (?) | -1.28% |
| Original Reserves | 74.13 | 12.4% | 9 | 0.77% |
| No Reserves | 270.39 | 46.5%(?) | 2 (?) | 2.79% |

Some things to note:

- Just changing the reserves can make a very large difference in the resulting profit measures.
- These scenarios show some of the flaws and limitations of some of these profit measures.
- When we hold no reserves, we have negative profitability in the later years, which isn't generally desired.

# Using Profit Testing to Set Reserves

We've seen how reserves impact the profit vectors and profit measures. (In general, lower reserves $\rightarrow$ higher profitability.)

We could then target a particular level of profitability or pattern of profits, and then set reserves accordingly to achieve the desired profits. In theory, we could target any sort of profit pattern.

As an example, let's consider trying to maximize our profits, subject to never having negative profitability (aside from the acquisition expenses).

- What pattern of profits would we want?
- How could we set reserves to meet this goal?

# Using Profit Testing to Set Reserves — Zeroization

We calculate the **zeroized** reserves by working backwards from the end of the contract, solving for the reserve that results in a zero emerging profit. If this procedure results in a negative value for ${}_kV^Z$, we set ${}_kV^Z = 0$. For this example we have:

$$
\begin{aligned}
{}_{10}V^Z &= 0 \\
Pr_{10} &= {}_9V^Z + P_{10} - E_{10} + I_{10} - EDB_{10} - E_{10}V \\
{}_9V^Z &= 353.45 \\
Pr_9 &= {}_8V^Z + P_9 - E_9 + I_9 - EDB_9 - E_9V \\
{}_8V^Z &= 587.66 \\
&\vdots \\
Pr_4 &= {}_3V^Z + P_4 - E_4 + I_4 - EDB_4 - E_4V \\
{}_3V^Z &= 247.62
\end{aligned}
$$

# Using Profit Testing to Set Reserves — Zeroization

$$Pr_3 = {}_2V^Z + P_3 - E_3 + I_3 - EDB_3 - E_3V$$

$${}_2V^Z = -78.17 \quad \Rightarrow \quad {}_2V^Z = 0$$

$$Pr_2 = {}_1V^Z + P_2 - E_2 + I_2 - EDB_2 - E_2V$$

$${}_1V^Z = -404.85 \quad \Rightarrow \quad {}_1V^Z = 0$$

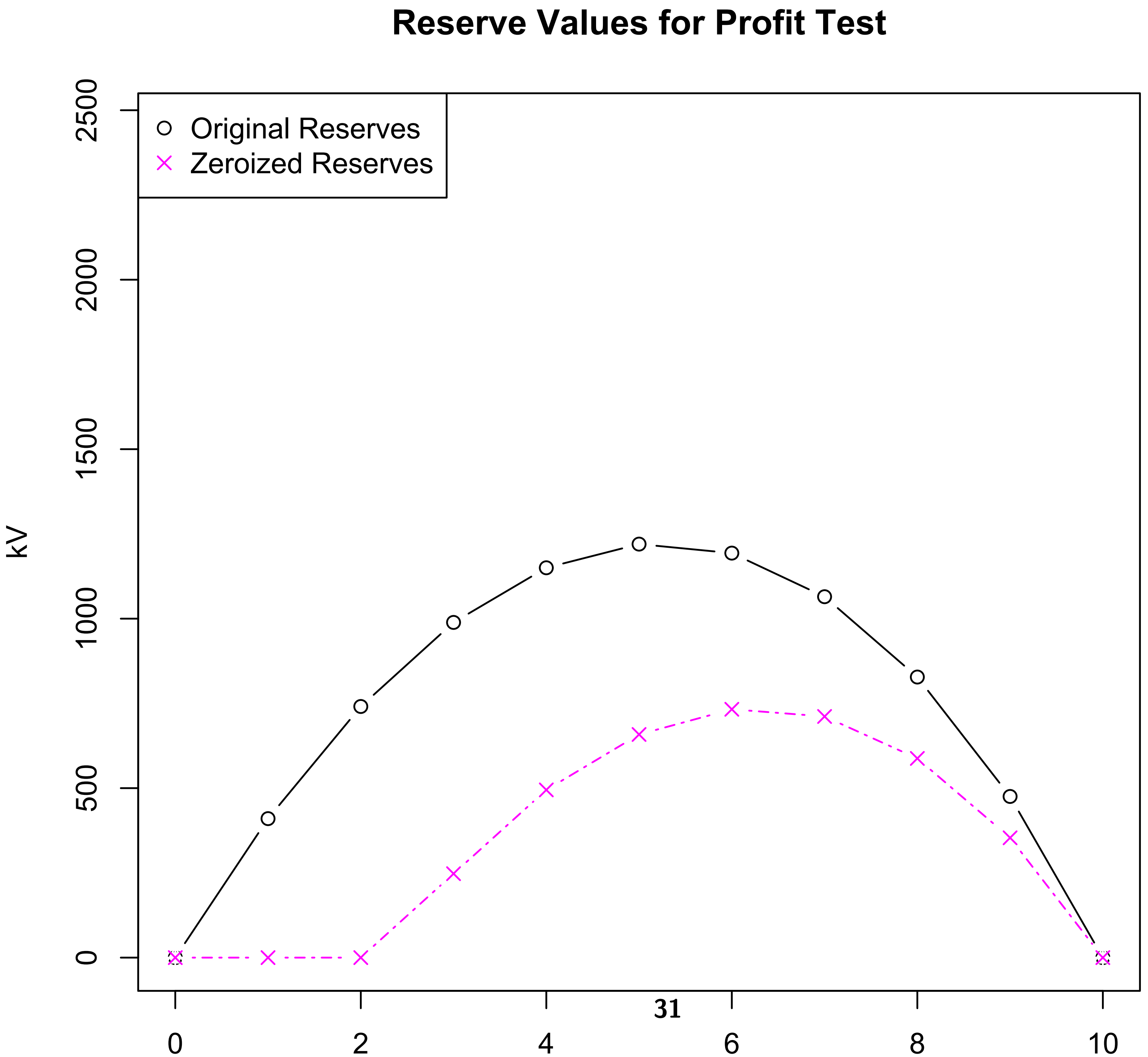
$$Pr_1 = {}_0V^Z + P_1 - E_1 + I_1 - EDB_1 - E_1V$$

$${}_0V^Z = -499.63 \quad \Rightarrow \quad {}_0V^Z = 0$$

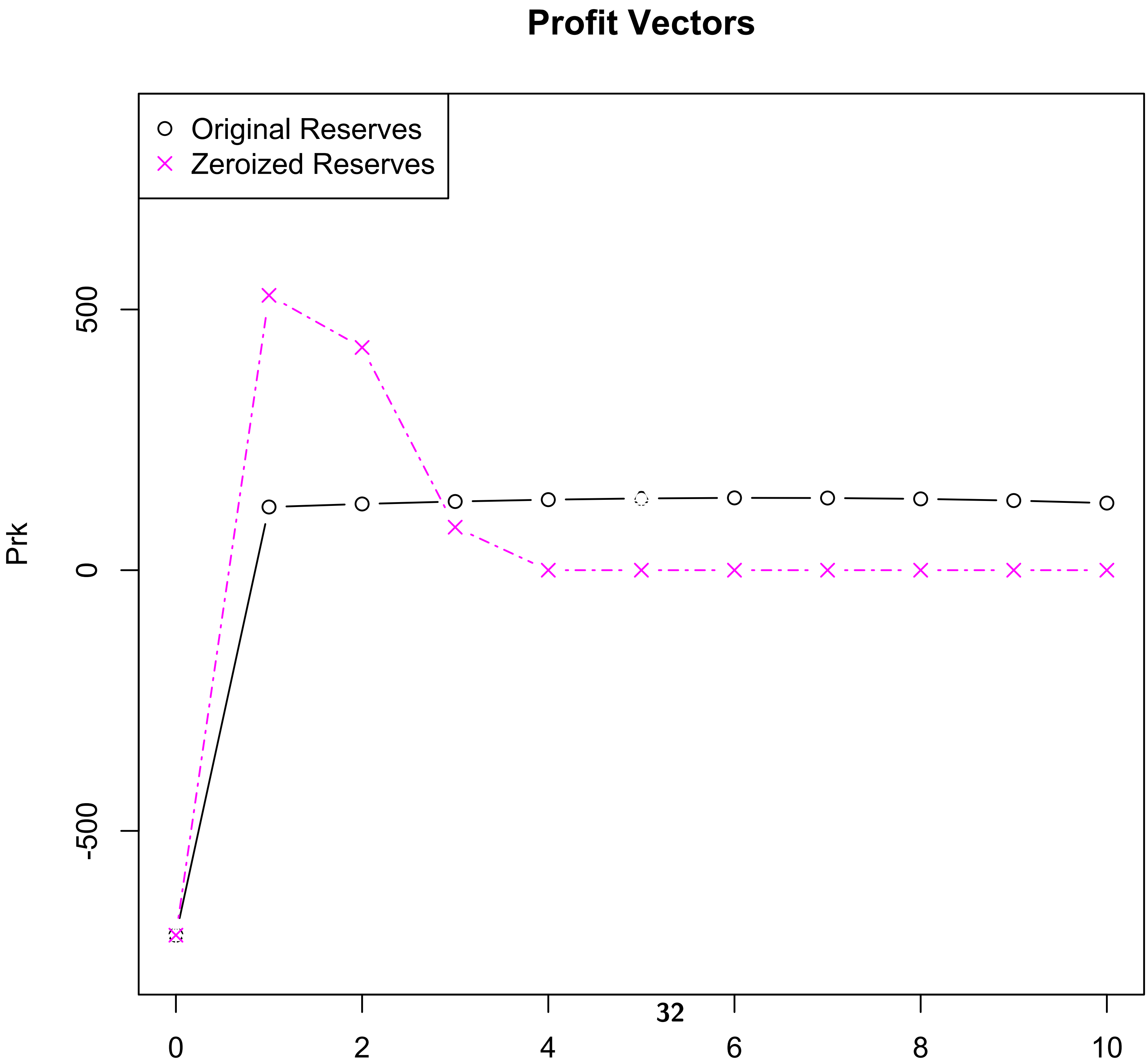
| $k$ | ${}_kV$ | $k$ | ${}_kV$ |
|---|---|---|---|
| 0 | 0.00 | 5 | 658.32 |
| 1 | 0.00 | 6 | 732.63 |
| 2 | 0.00 | 7 | 711.42 |
| 3 | 247.62 | 8 | 587.65 |
| 4 | 494.78 | 9 | 353.45 |

Table: Zeroized Reserves

# Using Profit Testing to Set Reserves — Zeroization



# Using Profit Testing to Set Reserves — Zeroization



We can see that the zeroized reserves result in better profitability (by all of our profit measures) than the original reserves, while avoiding any negative profits in the later years.

| Scenario | NPV | IRR | DPP | Profit Margin |
|---|---|---|---|---|
| Strengthened Reserves | -124.23 | 8.3% | $\infty$ | -1.28% |
| Original Reserves | 74.13 | 12.4% | 9 | 0.77% |
| Zeroized Reserves | 189.31 | 29.04% | 2 | 1.95% |
| No Reserves | 270.39 | 46.5% | 2 | 2.79% |

# Profit Testing for Multi-State Models

The general ideas for profit testing multi-state models are the same as those for traditional products.

However, we have to calculate a profit vectors for each state.

For example, $Pr_k^{(j)}$ represents the expected profit at time $k$ for a policy in force and in state $j$ at time $k-1$.

Then we use the transition probabilities ${}_kp_x^{ij}$ to calculate the profit signature from the multiple profit vectors. For example, if the insured is at state 0 at issue, then

$$\Pi_k = \sum_j {}_{k-1}p_x^{0j}\, Pr_k^{(j)}$$

Once we have the profit signature, we can compute the various profit measures as usual.